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ABSTRACT

This paper shows that partial differential equations may be a possible area of application of mathematical programming. The solution of Laplace's equation with Neumann's condition is shown to be a minimum cost network flow problem with cost proportional to the arc flow. An algorithm of solving minimum quadratic cost network flow is given.

I. Introduction:

The purpose of this paper is to present a new numerical method for solving Laplace's equation with Neumann's condition prescribed on the boundary. This method uses the idea in the theory of network flows which is a branch of mathematical programming usually considered to be unrelated to the subject of partial differential equations. For simplicity of exposition we shall discuss the case in two dimensions. The generalization to three dimensions is obvious.

Consider a simply-connected domain G in the plane whose boundary Γ is a simple closed curve. We are interested in finding the solution of

(1) $$\nabla^2 \phi = 0 \quad \text{in } G$$

with $\frac{\partial \phi}{\partial n}$ as prescribed on Γ, where $\frac{\partial \phi}{\partial n}$ is the normal derivative of ϕ considered to be positive if ϕ decreases its value across the boundary from the outside region to G. In order to have a unique solution, it is necessary and sufficient to prescribe $\frac{\partial \phi}{\partial n}$ on Γ such that the line integral

(2) $$\oint_{\Gamma} \frac{\partial \phi}{\partial n} \, ds = 0 .$$

This problem arises from many problems of mathematical physics; we shall only give the physical interpretation from the point of view of fluid mechanics. We consider this problem as that of incompressible irrotational flow and thus $\nabla \phi$ is the velocity of the flow where $\frac{\partial \phi}{\partial n}$ on Γ is influx or outflux of the liquid across the

boundary where (1) is the equation of continuity. Since there is no source or sink inside C, (2) expresses the condition that the total inflow must equal the total outflow. We shall call the parts of Γ for which $\frac{\partial \phi}{\partial n}$ are positive, line sources, and the part of Γ for which $\frac{\partial \phi}{\partial n}$ are negative, line sinks.

The usual numerical method is to replace differential equation (1) by difference equation with uniform or non-uniform grids, and then calculate the value of the function at the discrete points where the spacing between grid lines is determined by the degree of accuracy desired.

For a non-uniform rectangular grid (see Fig. 1), the difference equation for the Laplace's equation at point P becomes

$$(3) \qquad \frac{\phi_E - \phi_P}{h_E(h_E + h_W)} - \frac{\phi_P - \phi_W}{h_W(h_E + h_W)} + \frac{\phi_N - \phi_P}{h_N(h_N + h_S)} - \frac{\phi_P - \phi_S}{h_S(h_N + h_S)} = 0$$

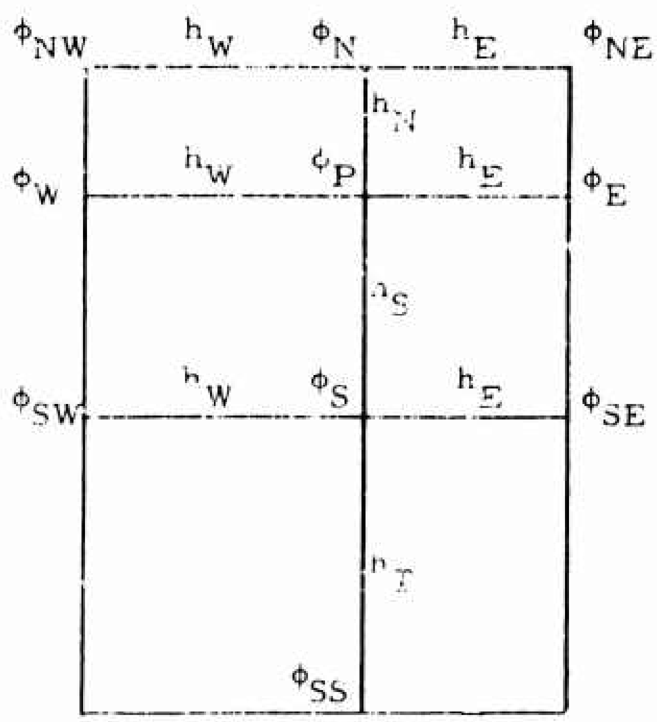

Fig. 1

Denoting $f_{EP} = \dfrac{\phi_E - \phi_P}{h_E(h_E + h_W)}$, etc., we can rewrite (3) as

$$(4) \qquad f_{EP} - f_{PW} + f_{NP} - f_{PS} = 0 \ .$$

If we consider the equation at the point S , using this notation, we also have

$$(5) \qquad f_{PS} = \frac{\phi_P - \phi_S}{h_S(h_S + h_T)} \ .$$

Since $h_N \neq h_T$ in general, the symbol f_{PS} etc., seems to be not uniquely defined. We shall first assume that all h's are equal and then discuss the non-uniform grid case later. For the case, all h's are equal, and the f_{ij} are just the differences between functions at points divided by $2h^2$ and are uniquely defined. Equation (4) suggests a strong analogy between the difference equation and the physical model of water flowing in pipes.

We shall now consider a physical model for the grid system to be a network. The boundary value problem of a network has already been discussed in Duffin [8] and Birckhoff and Diaz [2]. As usually done in the theory of network flows (see the most complete description by Ford and Fulkerson [9]) we consider a network which consists of nodes N_i and arcs B_{ij} connecting N_i and N_j . The value of the flow from N_i to N_j along the arc B_{ij} is denoted by f_{ij} and we have $f_{ij} = -f_{ji}$.

The conservation of liquid at a node N_i is expressed by the equation

$$(6) \qquad \sum_j f_{ij} - \sum_k f_{ki} = 0 \qquad \text{for all} \quad i \neq s, t,$$

where $f_{ij} \geq 0$ and $f_{ki} \geq 0$.

If a node N_s is a source node of strength v , we have

(7) $$\sum_j f_{sj} = v \;;$$

and for a sink node N_t of strength v , we have

(8) $$\sum_i f_{it} = v \;.$$

We have now replaced a continuous domain by a discrete network with line sources on Γ replaced by source nodes, and line sinks by sink nodes. The strength of a source representing a short line source γ is determined by

(9) $$v = \int_\gamma \frac{\partial \phi}{\partial n} \, ds$$

and similarly for a short line sink. In the following discussion, we shall assume that there is only one source node and one sink node.

The reader familiar with network flow problems or problems of operations research will suspect that this problem becomes a minimum cost flow problem [9] or that of Hitchcock transportation problem [12] with the sources as supplies and sinks as demands. But this is <u>not a standard minimum cost flow problem</u>, as shown later. For one thing, the network with infinity branch capacities can have many minimum cost flow patterns in the network if the cost is defined to be proportional to distance, where the solution of Laplace's equation is known to be unique.

II. Network Formulation:

To solve this question, we now turn to Dirchlet's principle (see the book by Courant [5]). The Dirchlet integral of a function g defined on G is given by

$$D[g] = \iint_G (g_x^2 + g_y^2)\,dx\,dy \,, \tag{10}$$

where g_x denotes the partial derivative of g with respect to x, similarly for g_y.

Now we quote from Courant [5].

Dirchlet's principle: Given a domain G whose boundary Γ consists of Jordan curves. Let g be a function continuous in $G + \Gamma$, piecewise smooth in G, and with finite Dirchlet integral $D[g]$. Consider the class of all functions ϕ continuous in $G + \Gamma$, piecewise smooth in G, and having the same boundary values as g. Then the problem of finding a function ϕ for which $D[\phi]$ attains a minimum d has a unique solution $\phi = u$. This function u is the solution of the boundary value problem for $\nabla^2 u = 0$ with the prescribed boundary value g on Γ.

Dirchlet's principle is certainly correct if in the boundary value problem, the normal derivative of g is prescribed instead of g. The finite analogy of (10) for a discrete network is then

$$D[\phi] = \frac{1}{2}\sum_{N,P} \frac{(h_W + h_E)h_N}{2} \cdot \left(\frac{\phi_N - \phi_P}{h_N}\right)^2 . \tag{11}$$

(The classical approach is to differentiate (11) with respect to ϕ_i and solve the simultaneous linear equations thus obtained.)

For the case all $h = 1$, we have

$$D[\phi] = D[f_{ij}] = \frac{1}{2} \sum_{i,j} f_{ij}^2 \quad . \tag{12}$$

So the network problem becomes that of minimizing (12) subjected to the condition (6) (7) (8) . This is a mathematical programming problem with linear constraints and non-linear objective function (12) . With the solution space being convex and the objective function quadratic (see for example [16]), relative minimum in the solution space implies absolute minimum. Furthermore, the solution is unique.

Instead of solving the problem as a quadratic programming one, we shall develop a network flow method. For convenience, we assume that a scale system is chosen such that the strength of N_s and of N_t are very large integers and "one" is the smallest positive constant.

Let us define the cost c_{ij} of shipping one unit of flow from N_i to N_j as

$$c_{ij} = \begin{cases} f_{ij} & \text{if } f_{ij} \geq 0 \\ -f_{ji} & \text{if } f_{ji} > 0 \quad . \end{cases} \tag{13}$$

This problem is now a minimum cost flow problem with the cost of shipping along an arc depending on the direction and the amount of flow in that arc.

We are interested in the flow pattern which satisfies the supply and demand conditions as defined by (6) (7) and (8) with the total cost

(14) $$D = \sum c_{ij} f_{ij} = \sum f_{ij}^2$$

a minimum. Let us call this minimum cost flow pattern the optimum flow pattern.

Clearly, in this optimum flow pattern, if a set of arcs $B_{i1}, B_{12}, \ldots, B_{mi}$ form a cycle, then we cannot have $f_{i1}, f_{12}, \ldots, f_{mi}$ all positive. Otherwise we can subtract from each arc the amount $\delta = \min(f_{i1}, f_{12}, \ldots, f_{mi}) > 0$, with (6) (7) and (8) still satisfied, but with the value of (14) reduced. This is no surprise as we know from the continuous case, the solution of Laplace's equation represents irrotational flow. This will be called the condition of no circulation.

Consider now the case for which there is a cycle formed by arcs in which we pick a node N_i and a node N_j. Let us consider the flows $f_{i1}, f_{12}, \ldots, f_{mj}$ in the first path and the flows $f_{i1'}, f_{1',2'}, \ldots, f_{m'j}$ in the second path. Note these arc flows may be positive or negative. Then we must have

(15) $$\sum c_{ij} = \sum c'_{ij} ,$$

where the left side represents the marginal cost of shipping one unit of flow along the first path, and the right-hand side represents the marginal cost of shipping along the second path. If (15) is not true, we can ship one unit along the cheaper path and go back on the more expensive path. This will make (6) (7) and (8) all satisfied but make (14) reduced.

We can now state the algorithm of constructing the optimum flow pattern. This algorithm is similar to the algorithm used in [3], [9] and [15] for constructing minimum cost flow where the cost is the same as the length of the arc or as its negative.

Let us define the cost of a path from N_s to N_t as the sum of costs of arcs travelling along that direction.

III. The Method and its Proof:

This algorithm can be described as follows. Assume that we have sent i units of flow from N_s to N_t $(i = 0, 1, \ldots, v)$.

Step 1: Find the cheapest path from N_s to N_t with the costs of arcs defined by (13). Let the path cost be C_{i+1} (i.e., the cost of the $(i+1)^{th}$ path).

Step 2: If $C_i = C_{i+1} \neq 0$, go to Step 3. If $C_i = C_{i+1} = 0$ or $C_i < C_{i+1}$, send one unit of flow from N_s to N_t along the cheapest path. Redefine the costs of arcs on the path. Go back to Step 1.

Step 3: Let f_{jk} be the arc flow from N_j to N_k, then $\phi_j - \phi_k = 2h^2 f_{jk} (\frac{v}{i})$.

There exist many papers about the algorithm of finding the cheapest path (or the shortest path). We shall only mention the first paper on this subject by Ford [9], the later modification by Dantzig [7], the matrix formulation by Berge [1], and the case of many sources and many sinks to the technique by Gomory and Hu [11].

We shall say N_i covers N_j if there is a positive flow f_{ij} from N_i to N_j. Since there is no circulation in the flow pattern, we can consider the set of all nodes to be partially ordered. For the network with its nodes partially ordered by its flow pattern we can also find the most expensive path (without cycles) from N_s to N_t (see for example [9]). For the optimum flow pattern, the costs of the most expensive path and the cheapest path should be

equal.

First we prove that this algorithm does give a flow pattern in which costs of all flow paths are equal. Assume that there are two paths P_1 and P_2 with the cost $C(P_1) < C(P_2)$. In sending the last unit flow along P_2, we are not using the cheapest path P_1 contradictory to the assumption that we always use the cheapest path.

Second, we prove that when costs of all paths are equal, the optimum solution is obtained. Assume that there is an optimum solution with the flow pattern different from that obtained by the algorithm. Assume the path cost of the pattern obtained by the algorithm is $C(h)$ and that of the optimum solution is $C(0)$.

Since it is necessary for an optimum solution to have all paths the same cost, if the optimum solution is not obtained by this algorithm, it must have all paths the same cost with $C(0) < C(h)$. Consider in the stage of the algorithm for which we have shipped $v - 1$ units of flows. Then we could send the last unit along a path with cost $C(0)$ and reduce the total cost. Since this, by assumption, is not possible, this means the Σf^h_{ij} in the path is greater than Σf^0_{ij} in that path and also for any part of the path. This means $f^h_{ij} > f^0_{ij}$. If this is true for all paths, since optimum solution gives v units, then by (7) this means we have shipped more than v units of flow, contradiction.

In order to speed up the convergence of this algorithm, lots of rules can be given for sending flows from the source to the sink when there are many arcs with $f_{ij} = 0$ in the arcs. We shall not

discuss such rules here.

The case of many sources of strength v_i and many sinks of strength v_k can be taken care of as usually done in network flows. Also, an upper bound on the number of steps to solve this problem can be given (see[15]).

IV. Examples:

Consider Fig. 2 where it is assumed that $f_{3,7} = 8$, $f_{11,15} = 8$ and $\frac{\partial\phi}{\partial n} = 0$ everywhere else, and $h = 1$.

According to the algorithm, we first send

Step 1: The cheapest path from N_3 to N_{15} costs zero. (As $f_{3,7} = f_{7,11} = f_{11,15} = 0$ initially.)

Step 2: Send one unit of flow along the path. Now $c_{3,7} = c_{7,11} = c_{11,15} = 1$ and $c_{7,3} = c_{11,7} = c_{15,11} = -1$.

Step 1: The cheapest path from N_3 to N_{15} costs now 2 as $c_{3,7} = 1$, $c_{6,7} = 0$, $c_{6,10} = 0$, $c_{10,11} = 0$, $c_{11,15} = 1$.

Step 2. $c_0 = 0 < c_1 = 2$. Send one unit of flow along the arc $B_{3,7}$, $B_{7,6}$, $B_{6,10}$, $B_{10,11}$ and $B_{11,15}$. Now $c_{3,7} = 2$, $c_{7,11} = 1$, $c_{11,15} = 2$, $c_{7,6} = 1$, $c_{6,10} = 1$, $c_{10,11} = 1$.

By this algorithm, we will send two more units along the path $B_{3,7}$, $B_{7,11}$, $B_{11,15}$ with $c_{3,7} = 4$, $c_{7,11} = 3$, $c_{11,15} = 4$. $c_{7,6} = 1$, $c_{6,10} = 1$, $c_{10,11} = 1$ with $C_4 = 8$

Then $C_4 = 8 = C_5 \neq 0$. We would go to Step 3 and obtain the differences between functions.

For example, $\phi_7 - \phi_{11} = 2 \cdot 1^2 \cdot 3 \cdot (\frac{8}{4}) = 12$.

The f_{ij} are shown in Fig. 3.

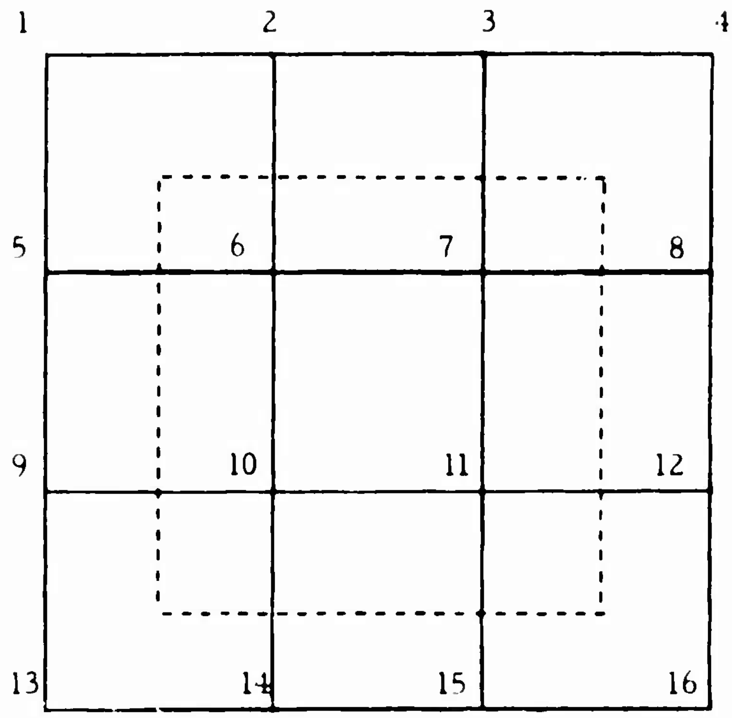

Fig . 2

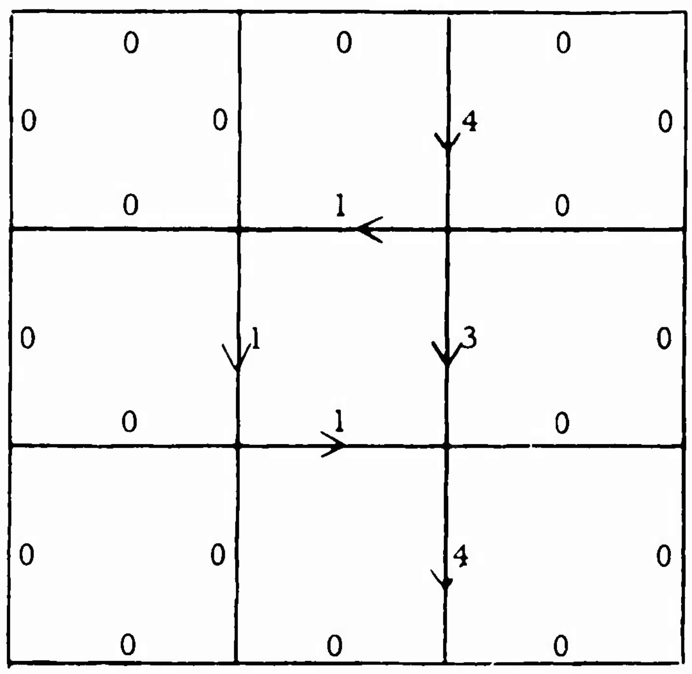

Fig. 3

Let us consider another example. A special example such that by very simple consideration, no iteration or solution of simultaneous equations is necessary.

Consider Fig. 4 where we assume that all normal derivatives are zero except that we have $\frac{\partial\phi}{\partial n} = 14$ at N_1 and $\frac{\partial\phi}{\partial n} = -14$ at $N_{1'}$. Due to the symmetry of this configuration, we can only consider the points N_1, N_2, N_3, N_4, N_5 and N_6.

Assume the amount of flow from N_4 to N_6 is x. Then from the consideration of flow at N_6 we have $f_{46} = f_{64'} = x$. Also, f_{45} and $f_{54'}$ are equal by symmetry. Furthermore, $f_{45} + f_{54} = f_{46} + f_{64}$, otherwise a reduction in cost can be achieved. Therefore $f_{45} = f_{54'} = x$. Clearly $f_{24} = f_{46} + f_{45} = 2x$, and from symmetry and the equal path cost condition $f_{23} = f_{35} = \frac{1}{2}(f_{24} + f_{45}) = \frac{3x}{2}$. This will make $f_{12} = f_{23} + f_{24} = \frac{3x}{2} + 2x = \frac{7x}{2}$. With $\frac{\partial\phi}{\partial n} = 14 = f_{12} + f_{12'} = 7x$. We let $x = 2$ and get the differences between all points as shown in Fig. 5.

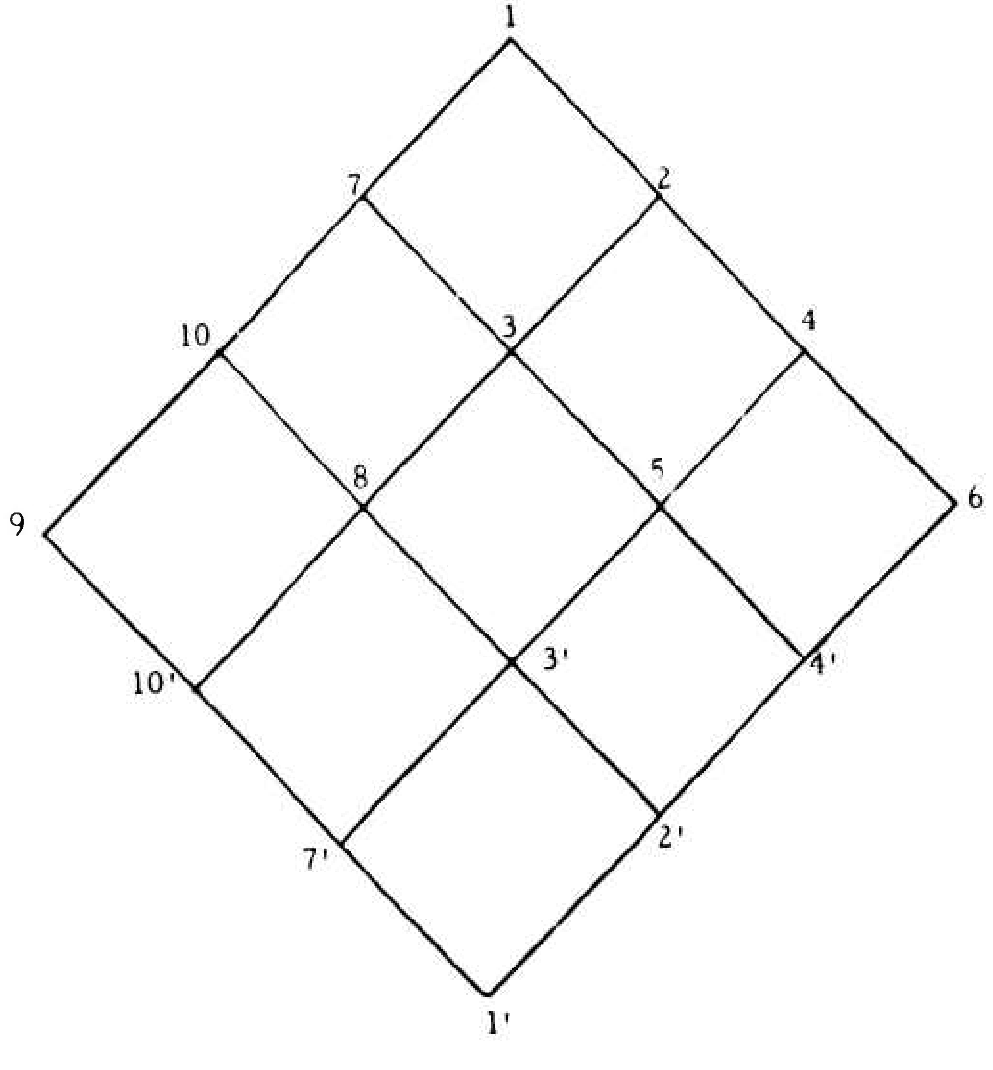

Fig. 4

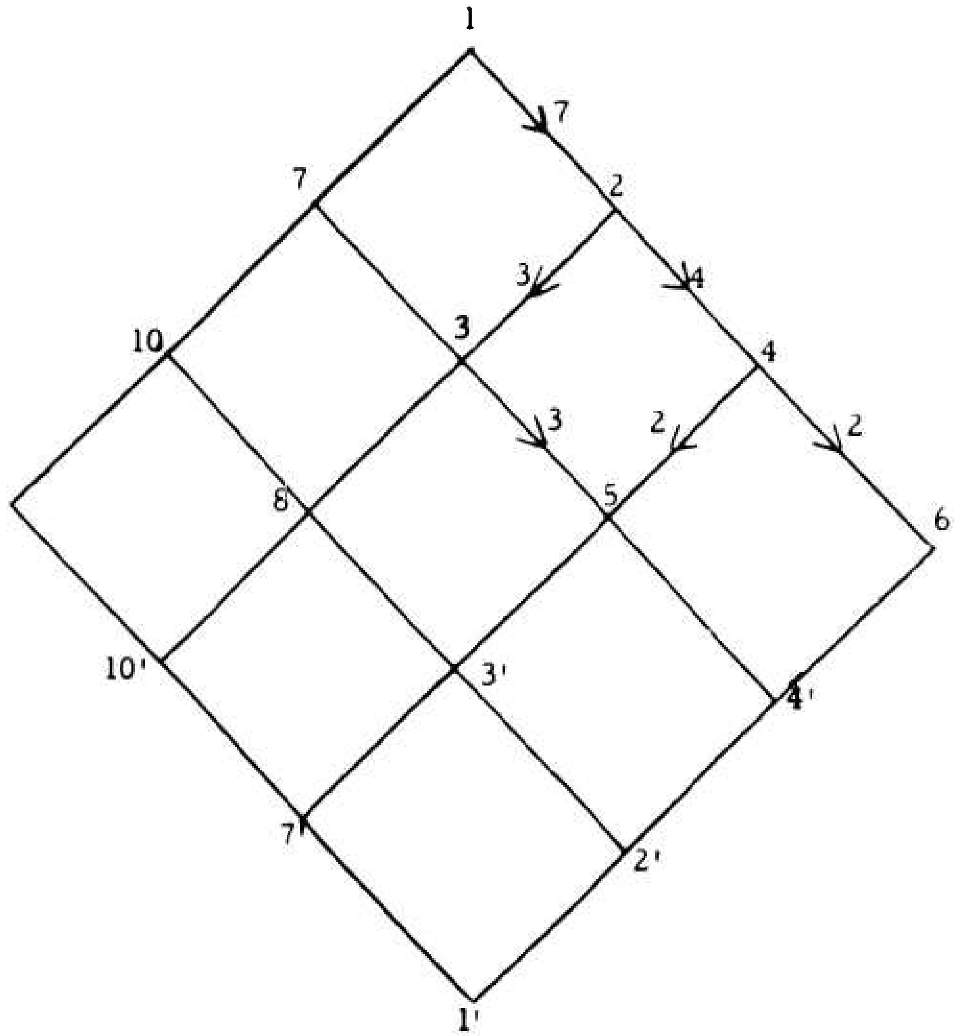

Fig. 5

V. Discussion

A few comments are in order.

1. This paper treats the Laplace's equation from the point of view of mathematical programming. The potentials at nodes are dual variables to the flows in arcs, and they play the same role as prices in the language of linear programming.

2. For Poisson's equation $\nabla^2\phi = v(x, y)$, the same interpretation of irrotational incompressible flow with sources and sinks on the boundary and inside the region G can be given. The Dirchlet integral (C) should be replaced by

$$\iint (\phi_x^2 + \phi_y^2 + 2v(x, y)\phi)\, dx\, dy \quad .$$

And the condition (2) on the boundary should be replaced by

$$\int_\Gamma \frac{\partial\phi}{\partial n}\, ds + \iint v(x, y)\, dx\, dy = 0$$

Although a variational formulation of a general elliptic partial differential equation of second order is possible, whether the method presented here is applicable is unknown.

3. For Dirichlet's condition prescribed on the boundary, we have just the same number of unknowns as the number of equations. The same is true for Neumann's condition if we fix ϕ at one point. If we introduce f_{ij} as variables, then we double the number of unknowns. The solution will not be unique unless we use the Dirchlet's integral as the objective function.

4. The network used in the theory of network flows is quite

arbitrary. Here the network is quite regular and there is no capacity constraints on the arcs. A better numerical method can be developed specially for this kind of network. Also the unimodular property (see for example [4] and [14]) of the incidence matrix describing the network gives certain insight to the network flow problem. This may play an important role in further development along the line of the numerical methods.

5. When the grids are not uniform, we have

$$f_{PS} = \frac{\phi_P - \phi_S}{h_S/(h_S + h_N)} \quad \text{or}$$

$$i_{PS} = \frac{\phi_P - \phi_S}{h_S/(h_S + h_T)} .$$

This makes the flow in an arc not unique. But this can be taken care of by introducing a multiplier on the arc from N_P to N_S. This multiplier depends only on the geometrical configuration of the grid. The paper by Jewell [14] on networks with gains can be applied. Right now, few papers have been written on networks with gains.

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